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Approximation by Linear Combinations of Positive Convolution Integrals

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DEDICATED TO THE MEMORY OF MY TEACHER EBERHARD L. STARK (1940–1986)

1. INTRODUCTION

Consider the singular convolution integral of $f \in C_{2\pi}$, namely,

$$I_n(\mathscr{p}; f; x) := (1/\pi) \int_{-\pi}^{\pi} f(x-t)\, p_n(t)\, dt \qquad (n \in \mathbb{N}; x \in \mathbb{R}), \tag{1.1}$$

where the kernel $\mathscr{p} = \{p_n\}_{n \in \mathbb{N}}$ is a sequence of even, normalized trigonometric polynomials (see Sect. 2 for the more precise notations). If $\mathscr{p}$ is positive, a well-known theorem of P. P. Korovkin [5, p. 88] states that there exists an arbitrarily often differentiable function $f \in C_{2\pi}$ such that

$$\limsup_{n \to \infty} n^2 \|I_n(\mathscr{p}; f; \cdot) - f(\cdot)\| > 0 \tag{1.2}$$

This result led to many investigations concerning the construction as well as the fine structure of kernels $\mathscr{p}$ for which the associated convolution integral $I_n(\mathscr{p}; f; x)$ has a better rate of approximation than $\mathcal{O}(n^{-2})$.

One method of construction is based on an extension of the foregoing Korovkin theorem: if $\mathscr{p}$ is of finite oscillation of degree 2κ, $\kappa \in \mathbb{N}_0$ (i.e., p_n has 2κ sign changes on $(-\pi, \pi)$ for each $n \in \mathbb{N}$), then the approximation rate is at most $\mathcal{O}(n^{-2\kappa-2})$ (see [6]). Thus one would try to multiply a positive kernel by a trigonometric polynomial having preassigned sign changes. For this approach see, e.g., [1, 7, 17, 18, 21] and for a historic overview see Stark [27].

In this paper, however, we shall consider linear combinations of positive kernels; thus of convolution integrals in order to improve the rate. This

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method has its roots in work of D. Jackson [19] and Ch. de la Vallée Poussin [29, pp. 43 ff.], just to mention two famous names. Nevertheless in these and later related papers only special cases were treated. The main aim here is to introduce a rather general method leading to the construction of linear combinations having high rates of approximation. If $\mathscr{p} = \{p_n\}_{n \in \mathbb{N}}$ is a positive kernel, we shall consider linear combinations $\mathscr{q} = \{q_n\}_{n \in \mathbb{N}}$ given by

$$q_n(x) := \sum_{v=1}^{s} \gamma_v p_{na_v}(x) \qquad (x \in \mathbb{R}) \tag{1.3}$$

with coefficients γ_v, the a_v being certain given naturals. For this purpose, the underlying kernel $\mathscr{p}$ needs first to be classified. Following E. L. Stark (cf. [25]), a positive kernel $\mathscr{p}$ is said to have the *expansion index* $\mu \in \mathbb{N}$, in short $\mathscr{p} \in S^{(\tau,\mu)}$, if the trigonometric moment of order 2σ, $\sigma \in \mathbb{N}$, given by (cf. Def. 2.3)

$$T(\mathscr{p}; 2\sigma; n) := \frac{1}{\pi} \int_{-\pi}^{\pi} \left(2 \sin \frac{t}{2}\right)^{2\sigma} p_n(t)\, dt \qquad (n \in \mathbb{N}),$$

is of the order

$$T(\mathscr{p}; 2\sigma; n) = \begin{cases} \mathcal{O}(n^{-\tau\sigma}), & 1 \leqslant \sigma \leqslant \mu, \\ \mathcal{O}(n^{-\tau(\mu+1/2)}), & u < \sigma, \end{cases} \tag{1.4}$$

either for $\tau = 1$ or $\tau = 2$. In fact, this classification will be equivalently introduced in terms of the so-called convergence factors, namely the Fourier-cosine coefficients, of the positive $\mathscr{p}$ (cf. (2.14)), since, following Stark [26] again, the convergence factors can be represented by a linear combination of these moments, and vice versa (cf. (2.12), (2.13)). The special case of Fejér-type kernels is treated in Section 3.

Since for positive $\mathscr{p} \in S^{(\tau,\mu)}$ one has

$$\|I_n(\mathscr{p}; \cos; \cdot) - \cos(\cdot)\| = \tfrac{1}{2} T(\mathscr{p}; 2; n) = \mathcal{O}(n^{-\tau}),$$

the trigonometric moments of order up to 2μ are all decreasing with rate higher than that of the second moment, namely $\mathcal{O}(n^{-\tau})$ by (1.4), which order already gives the maximal rate of approximation for convolution integrals with positive $\mathscr{p}$. This shows that it will be appropriate to determine s and the free coefficients γ_v in (1.3) such that for the trigonometric moments of the linear combination (1.3) there holds

$$T(\mathscr{q}; 2\sigma; n) = \begin{cases} \mathcal{O}(n^{-\tau s}), & 1 \leqslant \sigma \leqslant s, \\ o(n^{-\tau s}), & s < \sigma; \end{cases}$$

i.e., all moments up to the order $2s$ are all decreasing with the same rate. Thus if $\not{p}\in S^{(\tau,\mu)}$ with $2\leqslant\mu\in\mathbb{N}$, then for $2\leqslant s\leqslant\mu$ a linear combination (1.3) of $\not{p}$ may be constructed giving the rate $\mathcal{O}(n^{-\tau s})$ (instead of just $\mathcal{O}(n^{-\tau})$). This matter is dealt with in Section 4. Section 5 deals with two quantitative approximation theorems, those of Jackson and Voronovskaja-type, for $\not{q}$.

In Section 6 the construction described will be applied to several classical examples of positive $\not{p}$. Finally, Section 7 is devoted to a brief outline of the construction of linear combinations that are not saturated, a well-known example being the means of de La Vallée Poussin.

For analogous results in the case of approximation by Bernstein polynomials the reader is referred to, e.g., [2, 12]; for the situation of linear operator related to those of Bernstein see [15].

2. Notations and Preliminaries

In the following, let $\mathbb{N}$ be the set of naturals, $\mathbb{R}$ the set of real numbers, $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $C_{2\pi}$ the set of continuous and 2π-periodic functions on $\mathbb{R}$ with norm $\|f(\cdot)\|:=\sup_{x\in\mathbb{R}}|f(x)|$, $C_{2\pi}^{(r)}:=\{f\in C_{2\pi}\,|\,f^{(r)}\in C_{2\pi}\}$, $r\in\mathbb{N}$, and $C(\mathbb{R})$ the set of continuous functions on $\mathbb{R}$. The rth modulus of continuity, $r\in\mathbb{N}$, is defined for $f\in C_{2\pi}$ by $\omega_r(C_{2\pi};f;\delta):=\sup_{|h|\leqslant\delta}\|\sum_{k=0}^{r}(-1)^k\binom{r}{k}f(\cdot+kh)\|$, $\delta>0$; the Lipschitz classes are then $\mathrm{Lip}_r(C_{2\pi};\alpha):=\{f\in C_{2\pi}\,|\,\omega_r(C_{2\pi};f;\sigma)=\mathcal{O}(\delta^\alpha),\ \delta\to0+\}$, $0<\alpha\leqslant r\in\mathbb{N}$; the Landau symbols will always have to be understood for $n\to\infty$.

Let $\not{p}=\{p_n\}_{n\in\mathbb{N}}$ be a sequence of even trigonometric polynomials of degree at most $m(n)=\mathcal{O}(n)$, which are normalized by $(1/\pi)\int_{-\pi}^{\pi}p_n(t)\,dt=1$; i.e.,

$$p_n(x)=\frac{1}{2}+\sum_{k=1}^{m(n)}\rho_{k,n}(\not{p})\cos kx\qquad(x\in\mathbb{R}).\tag{2.1}$$

Here the Fourier- (cosine-) coefficients of $\not{p}$ are defined as usual by

$$\rho_{k,n}(\not{p}):=\begin{cases}(1/\pi)\int_{-\pi}^{\pi}p_n(t)\cos kt\,dt, & 0\leqslant k\leqslant m(n),\\ 0, & k>m(n).\end{cases}\tag{2.2}$$

In the following we shall consider for $f\in C_{2\pi}$ and a sequence $\not{p}$ given by (2.1) the (*singular*) *convolution integral*

$$I_n(\not{p};f;x):=(f*p_n)(x)\equiv\frac{1}{\pi}\int_{-\pi}^{\pi}f(x-t)\,p_n(t)\,dt\qquad(x\in\mathbb{R}).\tag{2.3}$$

In this context $\not{p}$ is referred to as a *polynomial even kernel* and the Fourier-coefficients as *convergence factors.*

The *Lebesgue constants* and thereby the operator norms of (2.3) are given by

$$L_n(\not{p}) := \frac{1}{\pi} \int_{-\pi}^{\pi} |p_n(t)| \, dt. \tag{2.4}$$

In order that (2.3) defines an *approximation process* on $C_{2\pi}$, i.e.,

$$\lim_{n \to \infty} \|I_n(\not{p}; f; \cdot) - f(\cdot)\| = 0 \qquad (f \in C_{2\pi}) \tag{2.5}$$

it is necessary and sufficient for the kernel $\not{p}$ to satisfy

$$\begin{aligned} &\text{(i)} \quad L_n(\not{p}) = \mathcal{O}(1), \\ &\text{(ii)} \quad \lim_{n \to \infty} \rho_{k,n}(\not{p}) = 1 \qquad (k \in \mathbb{N}). \end{aligned} \tag{2.6}$$

This is due to the well-known theorem of Banach and Steinhaus. In case the kernel $\not{p}$ is positive, i.e., $p_n(x) \geqslant 0$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, (2.6) reduces—in view of the Bohman–Korovkin theorem—to

$$\lim_{n \to \infty} \rho_{1,n}(\not{p}) = 1. \tag{2.7}$$

As an important technical tool the so-called *central factorial numbers* (cfn) of the first and second kind are needed (see, e.g., [22, p. 213]. To this end let, for $x \in \mathbb{R}$,

$$x^{[n]} := \begin{cases} x \prod_{i=1}^{n-1} \left(x + \frac{n}{2} - 1\right), & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases}$$

denote the *central factorial polynomial* of degree n.

DEFINITION 2.1. The *central factorial numbers* of the 1. kind t_k^n, respectively, the 2. kind T_k^n are the uniquely determined coefficients of the polynomials

$$\begin{aligned} x^{[n]} &= \sum_{k=0}^{n} t_k^n x^k \\ x^n &= \sum_{k=0}^{n} T_k^n x^{[k]}. \end{aligned} \qquad (n \in \mathbb{N}_0; x \in \mathbb{R})$$

Some properties of these numbers, needed in the following, are collected in

LEMMA 2.2.

$$\begin{aligned}
&\text{(i)} \quad t_0^n = T_0^n = \delta_{n,0} && (n \in \mathbb{N}_0),\\
&\text{(ii)} \quad t_k^n = T_k^n = 0 && (n < k),\\
&\text{(iii)} \quad t_{2k}^{2n+1} = t_{2k+1}^{2n} = T_{2k}^{2n+1} = T_{2k+1}^{2n} = 0 && (n, k \in \mathbb{N}_0), \qquad (2.8)\\
&\text{(iv)} \quad \sum_{k=0}^{\max\{n,m\}} t_k^n T_m^k = \sum_{k=0}^{\max\{n,m\}} T_k^n t_m^k = \delta_{n,m} && (n, m \in \mathbb{N}_0),\\
&\text{(v)} \quad T_k^n = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{k}{2} - j\right)^n && (0 \leqslant k \leqslant n \in \mathbb{N}_0).
\end{aligned}$$

Above, $\delta_{n,m}$ *is Kronecker's* δ.

Although these numbers occur especially in the older literature on interpolation and combinatorial theory, they have only received meager attention up to now as is pointed out in a footnote in [22, p. 213]. Nevertheless the Taylor coefficients of the powers of the trigonometric functions and their inverses can be expressed in terms of the cfn in closed (!) form. Just to mention one result, one has for $p \in \mathbb{N}$

$$(\sin x)^p = \sum_{k=0}^{\infty} (-1)^k \, 2^{2k} \frac{p!}{(p+2k)!} T_p^{p+2k} x^{p+2k} \qquad (x \in \mathbb{R}),$$

$$(\arcsin x)^p = \sum_{k=0}^{\infty} (-1)^k \, 2^{2k} \frac{p!}{(p+2k)!} t_p^{p+2k} x^{p+2k} \qquad (|x| < 1).$$

The interested reader is referred to [9, 10], the latter giving an application in the field of signal analysis.

Whereas condition (2.6) guarantees only the approximation property, an investigation of the order of approximation leads one to study the asymptotic behaviour of so-called moments of the kernel in question. In case the kernel is a trigonometric polynomial it will turn out to be appropriate to consider the so-called trigonometric moments.

DEFINITION 2.3. Let $\not{p}$ be a kernel as in (2.1). For $\sigma \in \mathbb{N}_0$

$$T(\not{p}; 2\sigma; n) := \frac{1}{\pi} \int_{-\pi}^{\pi} \left(2 \sin \frac{t}{2}\right)^{2\sigma} p_n(t)\, dt \qquad (2.9)$$

is called a *trigonometric moment* of order 2σ.

The *algebraic moment* of order 2σ, $\sigma \in \mathbb{N}_0$, is defined by

$$M(\not{p}; 2\sigma; n) = \frac{1}{\pi} \int_{-\pi}^{\pi} t^{2\sigma} p_n(t)\, dt. \tag{2.10}$$

Notice that because the kernel is positive the trigonometric as well as algebraic moments of odd order vanish.

Due to the inequality $t/\pi \leqslant \sin(t/2) \leqslant t/2$, $0 \leqslant t \leqslant \pi$, one deduces for positive kernels immediately the estimate

$$(2/\pi)^{2\sigma} M(\not{p}; 2\sigma; n) \leqslant T(\not{p}; 2\sigma; n) \leqslant M(\not{p}; 2\sigma; n) \qquad (\sigma \in \mathbb{N}_0). \tag{2.11}$$

Furthermore, by the well-known inverse formulas

$$\left(2 \sin \frac{t}{2}\right)^{2\sigma} = \binom{2\sigma}{\sigma} + 2 \sum_{k=1}^{\sigma} (-1)^k \binom{2\sigma}{\sigma - k} \cos kt \qquad (t \in \mathbb{R}),$$

$$\cos kt = 1 + \sum_{\sigma=1}^{k} (-1)^\sigma \left(2 \sin \frac{t}{2}\right)^{2\sigma} \frac{1}{(2\sigma)!} \prod_{i=0}^{\sigma-1} (k^2 - i^2) \qquad (t \in \mathbb{R}),$$

and the property (2.8)(v) of the central factorial numbers, the trigonometric moments can be expressed in terms of the convergence factors, and vice versa (see [26]). In fact,

$$T(\not{p}; 2\sigma; n) = 2 \sum_{k=1}^{\sigma} (-1)^{k+1} \binom{2\sigma}{\sigma - k} (1 - \rho_{k,n}(\not{p})) \qquad (\sigma \in \mathbb{N}), \tag{2.12}$$

$$1 - \rho_{k,n}(\not{p}) = \sum_{\sigma=1}^{k} (-1)^{\sigma+1} \frac{T(\not{p}; 2\sigma; n)}{(2\sigma)!} \sum_{l=1}^{\sigma} t_{2l}^{2\sigma} k^{2l} \qquad (0 < k \leqslant m(n)). \tag{2.13}$$

This reveals that the investigation of the asymptotic behaviour of the trigonometric moments can be reduced to the asymptotic expansion of the difference $1 - \rho_{k,n}(\not{p})$ in negative powers of n; thus in order to derive quantitative approximation theorems this means that (2.6)(ii) is to be replaced by an asymptotic expansion of $1 - \rho_{k,n}(\not{p})$.

For this purpose an appropriate classification of kernels $\not{p}$, which will be suitable for our linear combinations later on (see [25]), is given by

DEFINITION 2.4. Let $\tau = 1$ or $\tau = 2$. A kernel $\not{p}$ given by (2.1) is said to have the *expansion index* $\mu \in \mathbb{N}$, i.e., $\not{p} \in S^{(\tau,\mu)}$, if for all $k \in \mathbb{N}$ there holds an expansion

$$1-\rho_{k,n}(\not{p})=\sum_{j=1}^{\mu}(-1)^{j+1}\psi_j(k)n^{-\tau j}+\mathcal{O}(n^{-\tau(\mu+1/2)}), \tag{2.14}$$

$$\psi_j(k):=\sum_{i=1}^{j}c_{i,j}k^{2i}, \tag{2.15}$$

with certain $c_{i,j}\equiv c_{i,j}(\not{p})\in\mathbb{R}$. In case the expansion does not break off with the $\mathcal{O}$-term but continues indefinitely, we set $\mu=\infty$.

Most of the known kernels belong to a class $S^{(\tau,\mu)}$. Definition 2.4 leads with (2.12) and (2.13) to the following important equivalence assertion.

LEMMA 2.5. *Let $\tau=1$ or $\tau=2$, and $\mu\in\mathbb{N}$. The following assertions are equivalent for a kernel $\not{p}$:*

(i) $\not{p}\in S^{(\tau,\mu)}$,

(ii) $T(\not{p};2\sigma;n)$

$$=\begin{cases}(2\sigma)!\sum_{j=\sigma}^{\mu}(-1)^{j+\sigma}n^{-\tau j}\sum_{i=\sigma}^{j}c_{i,j}T_{2\sigma}^{2i}+\mathcal{O}(n^{-\tau(\mu+1/2)}), & 1\leqslant\sigma\leqslant\mu,\\ \mathcal{O}(n^{-\tau(\mu+1/2)}), & \mu<\sigma,\end{cases}$$

the $c_{i,j}$ being given as in Definition 2.4.

Proof. For the implication (i) $\to$ (ii) the reader is referred to [25]. The inverse implication runs as follows: using (2.13) one obtains from (ii) for $k\in\mathbb{N}$

$$1-\rho_{k,n}(\not{p})=\sum_{\sigma=1}^{k}\sum_{l=1}^{\sigma}t_{2l}^{2\sigma}k^{2l}\sum_{j=\sigma}^{\mu}(-1)^{j+1}n^{-\tau j}\sum_{i=\sigma}^{j}c_{i,j}T_{2\sigma}^{2i}+\mathcal{O}(n^{-\tau(\mu+1/2)})$$

$$=:S+\mathcal{O}(n^{-\tau(\mu+1/2)}).$$

Interchanging the order of summation thrice now leads to

$$S=\sum_{j=1}^{\mu}(-1)^{j+1}n^{-\tau j}\sum_{i=1}^{j}c_{i,j}\sum_{l=1}^{i}k^{2l}\sum_{\sigma=1}^{i}t_{2l}^{2\sigma}T_{2\sigma}^{2i}$$

$$=\sum_{j=1}^{\mu}(-1)^{j+1}n^{-\tau j}\sum_{i=1}^{j}c_{i,j}\sum_{l=1}^{i}k^{2l}\sum_{\sigma=0}^{i}t_{2l}^{2\sigma}T_{2\sigma}^{2i},$$

where in the last step use was made of (2.8)(ii). The "orthogonality relation" (2.8)(iv) finally gives

$$S=\sum_{j=1}^{\mu}(-1)^{j+1}n^{-\tau j}\sum_{i=1}^{j}c_{i,j}k^{2i};$$

thus $\not{h}\in S^{(\tau,\mu)}$.

3. Fejér-Type Kernels

An important role will be played by so-called *Fejér-type kernels*: Let $\chi\in L^1(\mathbb{R})$ be normalized by

$$\int_{-\infty}^{\infty}\chi(t)\,dt=2\pi. \tag{3.1}$$

With respect to (2.1) we shall only consider such χ which are even and have a Fourier transform $\chi^\wedge$ with compact support; i.e.,

$$\chi^\wedge(v):=\frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(t)\,e^{-ivt}\,dt=0 \qquad (|v|>T) \tag{3.2}$$

for some $T>0$. Then $\chi^*=\{\chi_n^*\}_{n\in\mathbb{N}}$ with

$$\chi_n^*(x):=\frac{1}{2}\sum_{k=-\infty}^{\infty}n\chi[n(x+2k\pi)] \qquad (x\in\mathbb{R}) \tag{3.3}$$

is called a kernel of *Fejér type*. The closed representation according to (2.1) is given by

$$\chi_n^*(x)=\frac{1}{2}+\sum_{k=1}^{[nT]}\chi^\wedge\left(\frac{k}{n}\right)\cos kx \qquad (x\in\mathbb{R}). \tag{3.4}$$

Notice that condition (3.2) is essential for χ^* to be a polynomial kernel.

Now with the Poisson formula (cf. [5, p. 123]) the singular convolution integral (2.3) with kernel χ may be represented as a convolution integral on the real line as

$$I_n(\chi^*;f;x)\equiv\frac{1}{\pi}\int_{-\pi}^{\pi}f(x-t)\,\chi_n^*(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}f\left(x-\frac{t}{n}\right)\chi(t)\,dt. \tag{3.5}$$

The above conditions placed upon χ guarantee that $I_n(\chi^*;f;x)$ defines an approximation process on $C_{2\pi}$ [5, p. 125]; furthermore, the right side of

(3.5) may be extended to an approximation process on $C(\mathbb{R})$, and thus with f being not necessarily periodic.

In view of Definition 2.4 the simple dependence of the convergence factors χ^* on n, expressed through $\rho_{k,n}(\chi^*) = \chi^\wedge(k/n)$, $0 \leqslant k \leqslant [nT]$, allows one to establish a simple criterion for χ^* to be an element of some class $S^{(\tau,\mu)}$.

LEMMA 3.1. *Let* χ *be given by* (3.1), (3.2), *and additionally let* $\chi^\wedge \in C^{(2\mu)}[-\delta, \delta] \cap C^{(2\mu+1)}[0, \delta]$ *for some* $\mu \in \mathbb{N}$, $\delta > 0$. *Then for the corresponding Fejér-type kernel* χ^* *given by* (3.3) *there holds*

$$\chi^* \in S^{(2,\mu)}.$$

Proof. With Taylor's theorem one has for $v \in (0, \delta)$

$$\chi^\wedge(v) = \sum_{j=0}^{2\mu} \frac{[\chi^\wedge]^{(j)}(0)}{j!} v^j + \frac{[\chi^\wedge]^{2\mu+1}(\theta v)}{(2\mu+1)!} v^{2\mu+1} \qquad (\theta \in (0, 1)).$$

Since χ is even and thus $\chi^\wedge$ as well, the derivatives of odd degree vanish and there remains (notice $\chi^\wedge(0) = 1$ with (3.1))

$$1 - \chi^\wedge(v) = -\sum_{j=1}^{\mu} \frac{[\chi^\wedge]^{2j}(0)}{(2j)!} v^2 j - \frac{[\chi^\wedge]^{(2\mu+1)}(\theta v)}{(2\mu+1)!} v^{2\mu+1}.$$

Setting $v = k/n$ with $k \in \mathbb{N}$ and with $n \to \infty$ one obtains, noting (3.4),

$$1 - \rho_{k,n}(\chi^*) = 1 - \chi^\wedge\left(\frac{k}{n}\right) = -\sum_{j=1}^{\mu} \frac{[\chi^\wedge]^{2j}(0)}{(2j)!} \left(\frac{k}{n}\right)^{2j} + \mathcal{O}(n^{-2(\mu+1/2)}),$$

giving $\chi^* \in S^{(2,\mu)}$.

We shall return to Fejér-type kernels in Section 7.

4. LINEAR COMBINATIONS OF POSITIVE KERNELS

As pointed out in the introduction, due to the well-known Theorem of P. P. Korovkin the rate of convergence of convolution integrals having a positive kernel cannot exceed $\mathcal{O}(n^{-2})$; i.e., there exist functions $f \in C_{2\pi}^\infty$ with

$$\limsup_{n \to \infty} n^2 \| I_n(\not{k}; f; \cdot) - f(\cdot) \| > 0.$$

The aim now is to raise the approximation order by considering linear combinations of suitable positive kernels.

The following elementary lemma, the proof of which follows from Cramer's theorem, is the underlying algebraic element for the subsequent construction.

LEMMA 4.1. *Let $s \in \mathbb{N}$ and $a_1 < a_2 < \cdots < a_s$ be s different natural numbers. The unique solution of the Vandermonde system of equations*

$$\sum_{\nu=1}^{s} \gamma_\nu a_\nu^{-\tau j} = \delta_{j,0} \qquad (j = 0, 1, ..., s-1) \tag{4.1}$$

is given by

$$\gamma_i = \frac{(-1)^{i+1}}{Q} \prod_{\substack{\nu=1 \\ \nu \neq i}}^{s} a_\nu^{-\tau} \prod_{\substack{1 \leqslant j < \nu \leqslant s \\ j, \nu \neq i}} (a_\nu^{-\tau} - a_j^{-\tau}) \qquad (i = 1, 2, ..., s), \tag{4.2}$$

where the system-determinant Q is given by

$$Q := \begin{vmatrix} 1 & \cdots & 1 \\ a_1^{-\tau} & \cdots & a_s^{-\tau} \\ \vdots & & \vdots \\ a_1^{-\tau(s-1)} & \cdots & a_s^{-\tau(s-1)} \end{vmatrix} \neq 0.$$

Furthermore, there holds

$$A_s := (-1)^{s+1} \sum_{\nu=1}^{s} \gamma_\nu a_\nu^{-\tau s} = \prod_{\nu=1}^{s} a_\nu^{-\tau}. \tag{4.3}$$

In the following we suppose $\not{k} \in S^{(\tau,\mu)}$, with $\tau = 1$ or $\tau = 2$, $\mu \in \mathbb{N}$, to be a positive kernel. For simplification we set

$$\beta_{n,s} := \begin{cases} n^{-\tau(s+1)}, & 1 \leqslant s < \mu, \\ n^{-\tau(\mu+1/2)}, & s = \mu. \end{cases} \tag{4.4}$$

Under the notations of Lemma 4.1 we consider linear combinations $\not{q} = \{q_n\}_{n \in \mathbb{N}}$, of even, trigonometric polynomials of degree $m(na_s)$, namely

$$q_n(x) := \sum_{\nu=1}^{s} \gamma_\nu p_{na_\nu}(x) \qquad (x \in \mathbb{R}). \tag{4.5}$$

THEOREM 4.2. *Let $\not{q}$ be a linear combination of a positive kernel $\not{k} \in S^{(\tau,\mu)}$ with $s \leqslant \mu$ as in (4.5). The following assertions hold:*

(i) *$\not{q}$ is a kernel according to (2.1);*

(ii) *the convergence factors associated with* φ *admit the expansion* (*see* (2.15), (4.3), (4.4))

$$1-\rho_{k,n}(\varphi)=n^{-\tau s}\psi_s(k)A_s+\mathcal{O}(\beta_{n,s}) \qquad (k\in\mathbb{N}); \tag{4.6}$$

(iii) *the trigonometric moments associated with* φ *admit the expansion* (*see* (2.15), (4.3), (4.4))

$$T(\varphi;2\sigma;n)=\begin{cases} n^{-\tau s}(-1)^{\sigma+1}(2\sigma)!A_s\sum_{i=\sigma}^{s}c_{i,s}T_{2\sigma}^{2i}+\mathcal{O}(\beta_{n,s}), & 1\leqslant\sigma\leqslant s,\\ \mathcal{O}(\beta_{n,s}), & s<\sigma. \end{cases} \tag{4.7}$$

Proof. It is obvious by the whole construction that (i) is valid. To prove (ii), one has by (4.5) and Definition 2.4,

$$\begin{aligned}\rho_{k,n}(\varphi)&=\sum_{v=1}^{s}\gamma_v\rho_{k,na_v}(\not{p})\\ &=\sum_{v=1}^{s}\gamma_v-\sum_{j=1}^{\mu}(-1)^{j+1}\psi_j(k)n^{-\tau j}\sum_{v=1}^{s}\gamma_v a_v^{-\tau j}+\mathcal{O}(\beta_{n,\mu}).\end{aligned}$$

The first sum equals 1 by (4.1) and in the second sum all summands vanish for $1\leqslant j\leqslant s-1$. Collecting all but the first non-vanishing term ($j=s$) into the $\mathcal{O}$-term, one then obtains

$$1-\rho_{k,n}(\varphi)=n^{-\tau s}\psi_s(k)(-1)^{s+1}\sum_{v=1}^{s}\gamma_v a_v^{-\tau s}+\mathcal{O}(\beta_{n,s}),$$

which is (4.6) with (4.3).

Concerning (iii), an insertion of expansion (4.6) into (2.12) gives with (2.15)

$$\begin{aligned}T(\varphi;2\sigma;n)&=2n^{-\tau s}A_s\sum_{k=1}^{\sigma}(-1)^{k+1}\binom{2\sigma}{\sigma-k}\psi_s(k)+\mathcal{O}(\beta_{n,s})\\ &=2n^{-\tau s}A_s\sum_{i=1}^{s}c_{i,s}\sum_{k=1}^{\sigma}(-1)^{k+1}\binom{2\sigma}{\sigma-k}k^{2i}+\mathcal{O}(\beta_{n,s}).\end{aligned}$$

Now, using (2.8)(v) yields (4.7).

Before turning to convergence assertions with rates it should be pointed out that the coefficients γ_i of the linear combination (4.5) only depend on the free parameters a_i and τ, the latter being given by the asymptotic properties of the kernel $\not{p}$; they do not depend on n!

Furthermore one notices that whereas the trigonometric moments of $\not{p}$ up to the order 2μ grow linearly, the moments of the linear combination $\not{q}$ up to the order $2s$ behave asymptotically all like $\mathcal{O}(n^{-\tau s})$.

5. Approximation Theorems

It is clear by the preceding results that a kernel given by (4.5) fulfills (2.6), and thus the corresponding convolution integral (2.3) defines an approximation process on $C_{2\pi}$. In order to consider quantitative approximation properties it is desirable to study theorems such as those of *Voronovskaja and Jackson type.*

5.1. *Voronovskaja-Type Theorem*

For positive kernels $\not{p} \in S^{(\tau,\mu)}$ with $\mu \in \mathbb{N}$ the *Voronovskaja-type theorem* states that (see, e.g., [4])

$$\lim_{n\to\infty} \|n^{\tau}\{I_n(\not{p};f;\cdot)-f(\cdot)\}-kf^{(2)}(\cdot)\|=0 \qquad (f\in C_{2\pi}^{(2)}), \tag{5.1}$$

with $k=k(\not{p})\in\mathbb{R}$. (Commonly a pointwise version of (5.1)—and thus a weaker version—is referred to in the literature as Voronovskaja-type theorem.)

The following Voronovskaja-type theorem will show how one can improve the order in (5.1) by introducing linear combinations (4.5) and their corresponding convolution integrals.

Theorem 5.1. *Let $\not{q}$ be a linear combination of a positive kernel $\not{p}\in S^{(\tau,\mu)}$ with $s\leqslant\mu$ as in* (4.5). *Then there holds for $f\in C_{2\pi}^{(2s)}$ the Voronovskaja-type expansion*

$$\lim_{n\to\infty} \|n^{\tau s}\{I_n(\not{q};f;\cdot)-f(\cdot)\}-A_s\sum_{k=1}^{s}(-1)^{k+1}c_{k,s}f^{(2k)}(\cdot)\|=0. \tag{5.2}$$

Proof. A mixed algebraic-trigonometric Taylor formula will be used (see, e.g., [28]): for $f\in C_{2\pi}^{(2s)}$ there holds $(x, t\in\mathbb{R})$

$$\begin{aligned} f(x+t)-f(x) &= \sum_{k=0}^{s-1}\frac{f^{(2k+1)}(x)}{(2k+1)!}t^{2k+1} \\ &\quad+\sum_{k=1}^{s}f^{(2k)}(x)\sum_{j=k}^{s}(-1)^{k+j}\frac{t_{2k}^{2j}}{(2j)!}\left(2\sin\frac{t}{2}\right)^{2j} \\ &\quad+R_s(f;x;t), \end{aligned} \tag{5.3}$$

the remainder term being given by

$$R_s(f;x;t):=\sum_{k=1}^{s}\frac{f^{(2k)}(x)}{(2k)!}\Theta_{k,s}(t)\,t^{2s+2}$$
$$+\frac{f^{(2s)}(\eta)-f^{(2s)}(x)}{(2s)!}\,t^{2s}. \tag{5.4}$$

Here $\Theta_{k,s}$ denotes a continuous function independent of f, and $\eta=\eta(f;x;t)$ lies between x and $x+t$.

The convolution of (5.3) with the kernel φ then gives (notice that the moments of odd order vanish, φ being even)

$$I_n(\varphi;f;x)-f(x)=\sum_{k=1}^{s}f^{(2k)}(x)\sum_{j=k}^{s}(-1)^{j+k}\frac{t_{2k}^{2j}}{(2j)!}T(\varphi;2j;n)$$
$$+\frac{1}{\pi}\int_{-\infty}^{\infty}q_n(t)\,R_s(f;x;t)\,dt=:S_1+S_2. \tag{5.5}$$

Now S_1 turns out to be, with (4.7),

$$S_1=n^{-\tau s}A_s\sum_{k=1}^{s}f^{(2k)}(x)\sum_{j=k}^{s}(-1)^{k+1}\,t_{2k}^{2j}\sum_{i=j}^{s}c_{i,s}T_{2j}^{2i}$$
$$+\mathcal{O}(\beta_{n,s})\sum_{k=1}^{s}f^{(2k)}(x).$$

Here and in the following the Landau symbols are always independent of f and x. Using (2.8)(iv) and the abbreviation $\alpha_{n,s}:=n^{\tau s}\beta_{n,s}$, one derives after some calculations

$$S_1=n^{-\tau s}\left\{A_s\sum_{k=1}^{s}(-1)^{k+1}c_{k,s}f^{(2k)}(x)+\mathcal{O}(\alpha_{n,s})\sum_{k=1}^{s}f^{(2k)}(x)\right\}. \tag{5.6}$$

Thus (5.5) and (5.6) lead to

$$n^{\tau s}\{I_n(\varphi;f;x)-f(x)\}-A_s\sum_{k=1}^{s}(-1)^{k+1}c_{k,s}f^{(2k)}(x)$$
$$=\mathcal{O}(\alpha_{n,s})\sum_{k=1}^{s}f^{(2k)}(x)+n^{\tau s}S_2. \tag{5.7}$$

Now one just needs to estimate the term S_2; in fact,

$$
\begin{aligned}
|S_2| &\leqslant \sum_{k=1}^{s} \frac{|f^{(2k)}(x)|}{(2k)!} \frac{1}{\pi} \int_{-\pi}^{\pi} t^{2s+2} |\Theta_{k,s}(t)\, q_n(t)|\, dt \\
&\quad + \frac{1}{(2s)!} \frac{1}{\pi} \int_{-\pi}^{\pi} |f^{(2s)}(\eta) - f^{(2s)}(x)|\, |q_n(t)|\, t^{2s}\, dt \\
&=: S_2^1 + S_2^2.
\end{aligned}
$$

Because of (2.10), (2.11), and (4.5) one obtains for S_2^1,

$$
\begin{aligned}
S_2^1 &\leqslant M(\varphi; 2s+2; n) \sum_{k=1}^{s} \frac{\|f^{(2k)}\|}{(2k)!} \|\Theta_{k,s}\| \\
&= \mathcal{O}(1)\, T(\varphi; 2s+2; n) \sum_{k=1}^{s} \|f^{(2k)}\|.
\end{aligned}
$$

With Lemma (2.5)(ii) this yields

$$
S_2^1 = n^{-\tau s} \mathcal{O}(\alpha_{n,s}) \sum_{k=1}^{s} \|f^{(2k)}\|. \tag{5.8}
$$

Concerning S_2^2, the difference under the integral can be estimated by the first modulus of continuity,

$$
\begin{aligned}
S_2^2 &\leqslant \frac{1}{(2s)!} \frac{2}{\pi} \int_0^{\pi} \omega(C_{2\pi}; f^{(2s)}; t)\, t^{2s} |q_n(t)|\, dt \\
&\leqslant \frac{1}{(2s)!} \sum_{\nu=1}^{s} |\gamma_\nu| \frac{2}{\pi} \int_0^{\pi} \omega(C_{2\pi}; f^{(2s)}; t)\, t^{2s} p_{na_\nu}(t)\, dt.
\end{aligned}
$$

Now with the inequality ($\delta > 0$)

$$
\begin{aligned}
\omega(C_{2\pi}; f^{(2s)}; t) &\leqslant \left(1 + \frac{t}{\delta}\right) \omega(C_{2\pi}; f^{(2s)}; \delta) \\
&\leqslant \left(2 + \frac{t^2}{\delta^2}\right) \omega(C_{2\pi}; f^{(2s)}; \delta)
\end{aligned}
$$

it follows with $\delta = \sqrt{\alpha_{n,s}}$, (2.11), and lemma (2.5)(ii),

$$
\begin{aligned}
S_2^2 &= \mathcal{O}(1) \left\{ \omega(C_{2\pi}; f^{(2s)}; \alpha_{n,s}^{1/2}) \sum_{\nu=1}^{s} T(\varphi; 2s; na_\nu) + \alpha_{n,s}^{-1} T(\varphi; 2s+2; na_\nu) \right\} \\
&= \mathcal{O}(n^{-\tau s})\, \omega(C_{2\pi}; f^{(2s)}; \alpha_{n,s}^{1/2}).
\end{aligned} \tag{5.9}
$$

Altogether we then have

$$\left\| n^{\tau s}\{I_n(\varphi; f; \cdot) - f(\cdot)\} - A_s \sum_{k=1}^{s} (-1)^{k+1} c_{k,s} f^{(2k)}(\cdot) \right\|$$

$$= \mathcal{O}(\alpha_{n,s}) \sum_{k=1}^{s} \|f^{(2k)}\| + \mathcal{O}(1)\, \omega(C_{2\pi}; f^{(2s)}; \alpha_{n,s}^{1/2}), \tag{5.10}$$

which implies (5.2) since $\alpha_{n,s} \to 0$ for $n \to \infty$ by (4.4).

5.2. *Jackson-Type Theorem*

The well-known *Jackson-type theorem* for positive kernels $\not{k}$ states that the rate of approximation can be estimated by the second modulus of continuity of the underlying function $f \in C_{2\pi}$ (see, e.g., [5, p. 69]; thus

$$\|I_n(\not{k}; f, \cdot) - f(\cdot)\| = \mathcal{O}(1)\, \omega_2(C_{2\pi}; f; \sqrt{1 - \rho_{1,n}(\not{k})})$$
$$= \mathcal{O}(1)\, \omega_2(C_{2\pi}; f; \sqrt{T(\not{k}; 2; n)}). \tag{5.11}$$

In case $\not{k} \in S^{(\tau,\mu)}$, it follows immediately that

$$\|I_n(\not{k}; f; \cdot) - f(\cdot)\| = \mathcal{O}(1)\, \omega_2(C_{2\pi}; f; n^{-\tau/2}). \tag{5.12}$$

Before formulating a Jackson-type theorem for the presented linear combinations we need a further basic lemma:

LEMMA 5.2. *Let $j, k \in \mathbb{N}$ with $1 \leqslant j \leqslant k$, $f \in C_{2\pi}^{(k)}$. Then*

$$\|f^{(j)}\| \leqslant (2\pi)^{k-j}\, \|f^{(k)}\|. \tag{5.13}$$

Proof. The case $j = k$ is trivial. If $1 \leqslant j < k$, then

$$\int_{-\pi}^{\pi} f^{(j+1)}(u)\, du = f^{(j)}(\pi) - f^{(j)}(-\pi) = 0.$$

Thus there exists $\xi \in (-\pi, \pi)$ with $f^{(j+1)}(\xi) = 0$ and so

$$|f^{(j)}(x)| = \left| \int_{\xi}^{x} f^{(j+1)}(u)\, du \right| \leqslant |x - \xi|\, \|f^{(j+1)}\| \leqslant 2\pi \|f^{(j+1)}\|.$$

Iterative repetition delivers (5.13).

Now to the Jackson-type theorem.

THEOREM 5.3. *Let φ be a linear combination of a positive kernel $\not{k} \in S^{(\tau,\mu)}$ with $s \leqslant \mu$ as in (4.5). There holds for $f \in C_{2\pi}$*

$$\|I_n(\varphi; f; \cdot) - f(\cdot)\| = \mathcal{O}(1)\, \omega_{2s}(C_{2\pi}; f; n^{-\tau/2}). \tag{5.14}$$

Proof. According to a general result of Butzer and Scherer, it suffices to prove [3, 4] a *stability property* as well as a *Jackson-type inequality*,

$$L_n(\varphi) = \mathcal{O}(1) \tag{5.15}$$

$$\|I_n(\varphi; f; \cdot) - f(\cdot)\| = \mathcal{O}(1) n^{-\tau s} \|f^{(2s)}\| \qquad (f \in C_{2\pi}^{(2s)}). \tag{5.16}$$

Since (5.15) has already been proved it remains to establish (5.16). But this follows easily from (5.10), using (5.13).

COROLLARY 5.4. *If* $f \in \mathrm{Lip}_{2s}(C_{2\pi}, \alpha)$, $0 < \alpha \leqslant 2s$, *then*

$$\|I_n(\varphi; f; \cdot) - f(\cdot)\| = \mathcal{O}(n^{-\tau\alpha/2}). \tag{5.17}$$

Thus for suitable kernels $\varphi \in S^{(\tau, \mu)}$ with $\mu \geqslant 2$, the rate of convergence can, in comparison with (5.12), be improved from $\mathcal{O}(n^{-\tau})$ to $\mathcal{O}(n^{-\tau s})$, $2 \leqslant s \leqslant \mu$.

6. EXAMPLES

6.1. *Jackson Kernel*

For $2 \leqslant p \in \mathbb{N}$ the positive *Jackson kernels* $\mathcal{J}^p = \{J_n^p\}_{n \in \mathbb{N}}$ are given in closed representation by

$$J_n^p(n) = \frac{1}{\lambda_{0,n}(p)} \left[\frac{\sin nx/2}{\sin x/2}\right]^{2p} \in \Pi_{np-p}^{+} \qquad (x \in \mathbb{R}), \tag{6.1}$$

where

$$\lambda_{k,n}(p) := \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{\sin nt/2}{\sin t/2}\right]^{2p} \cos kt \, dt \qquad (0 \leqslant k \leqslant np - p). \tag{6.2}$$

Thus the corresponding convergence factors are, according to (2.2),

$$\rho_{k,n}(\mathcal{J}^p) = \frac{\lambda_{k,n}(p)}{\lambda_{0,n}(p)} \qquad (0 \leqslant k \leqslant np - n). \tag{6.3}$$

The Jackson kernels are not of Fejér type. In [14] one finds the representation for $\lambda_{k,n}(p)$

$$\lambda_{k,n}(p) = \sum_{j=0}^{2p} (-1)^j \binom{2p}{j} \binom{np - nj + p - k - 1}{2p - 1} \qquad (0 \leqslant k \leqslant np - p). \tag{6.4}$$

With (2.8)(ii, v) one obtains after elaborate calculations for $k=0$,

$$\lambda_{0,n}(p)=\frac{1}{(2p-1)!}\sum_{i=1}^{p} n^{2i-1}t_{2i}^{2p}\sum_{j=0}^{p}(-1)^j\binom{2p}{j}(p-j)^{2i-1}, \tag{6.5}$$

and for $1\leqslant k\leqslant n-p$

$$\begin{aligned}\lambda_{k,n}(p)=&\frac{1}{(2p-1)!}\sum_{i=2}^{p} t_{2i}^{2p}\sum_{m=1}^{i-1}\binom{2i-1}{2m}n^{2i-2m-1}k^{2m}\\ &\times\sum_{j=0}^{p}(-1)^j\binom{2p}{j}(p-j)^{2i-2m-1}\\ &+\frac{(-1)^p}{(2p-1)!}\binom{2p}{p}\sum_{i=1}^{p}t_{2i}^{2p}k^{2i-1}+\lambda_{0,n}.\end{aligned} \tag{6.6}$$

Thus polynomial division of (6.6) by (6.5) delivers, with certain $c_{i,j}=c_{i,j}(\mathscr{J}^p)$ and $d_{k,p}\neq 0$,

$$1-\rho_{k,n}(\mathscr{J}^p)=\sum_{j=1}^{p-1}n^{-2j}\sum_{i=1}^{j}c_{i,j}k^{2i}+d_{k,p}n^{-2(p-1/2)}+o(n^{-2p}), \tag{6.7}$$

which reveals that $\mathscr{J}^p\in S^{(2,p-1)}$ and $\mathscr{J}^p\notin S^{(2,p)}$.

Therefore in order to achieve an approximation rate higher than $\mathcal{O}(n^{-2})$ with a linear combination (4.5) of Jackson kernels the parameter p has to be $p\geqslant 3$.

The special case of linear combinations (4.5) with $a_i=i$ is carried out in [20]. Nevertheless that author uses algebraic moments and does not notice that those of odd degree vanish, which then leads to combinations with twice the number of summations, when compared with our results.

6.2. *Jackson–de La Vallée Poussin Kernel*

Let us now consider a Fejér-type kernel . First let us define the so called *central B-splines* as the Fourier transforms of the powers of the sinc function ($\operatorname{sinc} x:=\sin x/x$, $x\in\mathbb{R}\backslash\{0\}$, $\operatorname{sinc} 0:=1$), i.e.,

$$B_m(v):=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\operatorname{sinc}\frac{t}{2}\right)^m\cos vt\,dt \qquad (2\leqslant m\in\mathbb{N},\ v\in\mathbb{R}). \tag{6.8}$$

Their closed form is given by (cf. [11])

$$B_m(v)=\begin{cases}\dfrac{1}{(m-1)!}\displaystyle\sum_{j=0}^{[(m/2)-|v|]}(-1)^j\binom{m}{j}\left(\frac{m}{2}-|v|-j\right)^{m-1}, & |v|\leqslant\dfrac{m}{2}\\ 0, & |v|>\dfrac{m}{2}.\end{cases} \tag{6.9}$$

For a historical overview the reader is referred to [8]. The main properties of B-splines are (cf. [23, p. 11; 24, p. 134])

$$\begin{aligned} B_m(v) &\geqslant 0 && (v \in \mathbb{R}), \\ B_m &\in C^{m-2}(\mathbb{R}) && (m \geqslant 2), \\ B_m &\in C^{m-1}\left[-\frac{m}{2}+i,\ -\frac{m}{2}+i+1\right] && (0 \leqslant i \leqslant m-1,\ i \in \mathbb{N}). \end{aligned} \tag{6.10}$$

Thus the function χ, given by

$$\chi(x) \equiv \chi_p(x) = 1/B_{2p}(0)\left(\operatorname{sinc}\frac{x}{2}\right)^{2p} \qquad (x \in \mathbb{R},\ p \in \mathbb{N}), \tag{6.11}$$

satisfies (3.1) and (3.2). Then for $p \in \mathbb{N}$ the corresponding Fejér-type kernel (3.3), namely, the kernel of *Jackson and de La Vallée Poussin* $\mathscr{P}^p = \{P_n^p\}_{n \in \mathbb{N}}$, is given by, according to (2.1),

$$P_n^p(x) = \frac{1}{2} + \sum_{k=1}^{np-1} \frac{B_{2p}(k/n)}{B_{2p}(0)} \cos kx \qquad (x \in \mathbb{R}). \tag{6.12}$$

For $p = 1$ this is the well-known *Fejér kernel* $\mathscr{F} = \{F_n\}_{n \in \mathbb{N}}$ with

$$F_n(x) = \frac{1}{2n}\left[\frac{\sin(nx|2)}{\sin x/2}\right]^2 \qquad (x \in \mathbb{R}) \tag{6.13}$$

and convergence factors

$$\rho_{k,n}(\mathscr{F}) = 1 - \frac{k}{n} \qquad (0 \leqslant k \leqslant k-1). \tag{6.14}$$

For $p = 2$ it is the classical kernel of *Jackson and de La Vallée Poussin* with

$$P_n^2(x) = \frac{2 + \cos x}{4n^3}\left[\frac{\sin(nx/2)}{\sin x/2}\right]^4 \qquad (x \in \mathbb{R}).$$

The convergence factors in this case are

$$\rho_{k,n}(\mathscr{P}^2) = \begin{cases} 1 - \dfrac{3}{2}\left(\dfrac{k}{n}\right)^2 + \dfrac{3}{4}\left(\dfrac{k}{n}\right)^3, & 0 \leqslant k \leqslant n-1, \\[2ex] \dfrac{1}{4}\left(2 - \dfrac{k}{n}\right)^3, & 0 \leqslant k \leqslant 2n-1. \end{cases} \tag{6.15}$$

According to Lemma 3.1 and (6.10) it follows that $\mathscr{P}^p \in S^{(2,p-1)}$. Using

(6.9) the closed representation of the convergence factors reads for $0 \leqslant k \leqslant n$,

$$
\begin{aligned}
1 - \rho_{k,n}(\mathscr{P}^p) &= \frac{1}{(2p-1)!B_{2p}(0)} \left\{ \sum_{i=1}^{p-1} \left(\frac{k}{n}\right)^{2i} \binom{2p-1}{2i} \sum_{j=1}^{p} (-1)^{j+p+1} \binom{2p}{p-j} j^{2p-2i-1} \right. \\
&\quad \left. + n^{-2(p-1/2)} \frac{(-1)^{p+1} k^{2p-1}}{2} \binom{2p}{p} \right\}, \qquad (6.16)
\end{aligned}
$$

which also shows $\mathscr{P}^p \notin S^{(2,p)}$. With (3.5) one gets the commonly known representation of the singular integral of Jackson and de La Vallée Poussin on the real line, namely

$$
I_n(\mathscr{P}^p; f; x) = B_{2p}^{-1}(0) \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x-t) \left(\operatorname{sinc} \frac{t}{2} \right)^{2p} dt \qquad (6.17)
$$

for $f \in C_{2\pi}$, $x \in \mathbb{R}$.

Following (3.5), a linear combination φ of Jackson–de La Vallée Poussin kernels $\mathscr{P}^p$, $p \geqslant 3$, leads with Theorem 5.1 to the following

COROLLARY 6.1. *Let φ be a linear combination of Jackson–de La Vallée Poussin kernels given by* (4.5) *with $s = p - 1$. There holds for $f \in C_{2\pi}^{(2p-2)}$ the Voronovskaja-type expansion*

$$
\begin{aligned}
\lim_{n \to \infty} \Bigg\| n^{2p-2} \{ I_n(\varphi; f; \cdot) - f(\cdot) \} \\
- \frac{A_{p-1}}{B_{2p}(0)} \frac{1}{2(2p-2)!} \binom{2p}{p} f^{(2p-2)}(\cdot) \Bigg\| = 0. \qquad (6.18)
\end{aligned}
$$

Finally, it should be mentioned that linear combinations (4.5) of Jackson–de La Vallée Poussin kernels in the particular case $a_i = 2^i$, $1 \leqslant i \leqslant s$, were already given in [29] (cf. as well [13]). Furthermore, in this respect the work [19] of D. Jackson is a milestone; he already uses linear combinations to prove his famous theorem (i.e., for $f \in C_{2\pi}^{(k-1)}$, $k \in \mathbb{N}$, and $f^{(k-1)} \in \mathrm{Lip}_1(C_{2\pi}; 1)$) that there exists a trigonometric polynomial t_n of degree n such that $\|f(\cdot) - t_n(\cdot)\| = \mathcal{O}(n^{-k})$. Nevertheless, these linear combinations are in general not trigonometric polynomials.

6.3. *de La Vallée Poussin Kernel*

The kernel of *de La Vallée Poussin* $\mathscr{V} = \{V_n\}_{n \in \mathbb{N}}$ is given by

$$
V_n(x) := \frac{(n!)^2}{2(2n)!} \left(2 \cos \frac{x}{2} \right)^{2n} \qquad (x \in \mathbb{R}, n \in \mathbb{N}), \qquad (6.19)
$$

the corresponding convergence factors being

$$\rho_{k,n}(\mathscr{V}) = \binom{2n}{n+k} \Big/ \binom{2n}{n} \qquad (0 \leqslant k \leqslant n). \tag{6.20}$$

Since (6.20) can be regarded as a (normalized) binomial distribution, it follows (cf. [16 p. 85]) that (6.20) admits the expansion

$$1 - \rho_{k,n}(\mathscr{V}) = k^2 n^{-1} - \tfrac{1}{2}(k^4 + k^2) n^{-2} + \tfrac{1}{3}(k^6 + 4k^4 + k^2) n^{-3} + \cdots. \tag{6.21}$$

This gives $\mathscr{V} \in S^{(1,\infty)}$, so that the kernel of de La Vallée Poussin is suitable for linear combinations (4.5) and this with arbitrary $s \in \mathbb{N}$. Nevertheless since $\tau = 1$ one only gets the maximal approximation rate $\mathcal{O}(n^{-s})$ instead of $\mathcal{O}(n^{-2s})$ for $f \in C_{2\pi}^{(2s)}$.

6.4. *Fejér–Korovkin Kernel*

The kernel of *Fejér and Korovkin* $\mathscr{K} = \{K_n\}_{n \in \mathbb{N}}$ together with its convergence factors is given by

$$K_n(x) = \frac{1}{n} \sin^2 \frac{\pi}{n} \frac{\cos^2 nx/2}{(\cos x - \cos \pi/n)^2} \qquad (x \in \mathbb{R}). \tag{6.22}$$

$$\rho_{k,n}(\mathscr{K}) = \left(1 - \frac{k}{n}\right) \cos \frac{k\pi}{n} + \frac{1}{n} \cot \frac{\pi}{n} \sin \frac{k\pi}{n} \qquad (1 \leqslant k \leqslant n-2). \tag{6.23}$$

An expansion of (6.23) gives

$$1 - \rho_{k,n}(\mathscr{K}) = \frac{\pi^2}{2} k^2 n^{-2} - \frac{1}{3} \pi^2 (k^3 - k) n^{-3} + \mathcal{O}(n^{-4}), \tag{6.24}$$

which shows that $\mathscr{K} \in S^{(2,1)}$ and $\mathscr{K} \notin S^{(2,\mu)}$, $\mu \geqslant 2$. Thus the kernel of Fejér and Korovkin is not suitable for building up linear combinations according to (4.5).

6.5. *Bohman–Zheng Weixing Kernel*

A kernel which is quite similar in many respects to the above kernel of Fejér and Korovkin is the kernel of *Bohman and Zheng Weixing* $\mathscr{Z} = \{Z_n\}_{2 \leqslant n \in \mathbb{N}}$. This one is of Fejér type and its convergence factors are given by (cf. [14, p. 70])

$$\rho_{k,n}(\mathscr{Z}) = \left(1 - \frac{k}{n}\right) \cos \frac{k\pi}{n} + \frac{1}{\pi} \sin \frac{k\pi}{n} \qquad (1 \leqslant k \leqslant n-1). \tag{6.25}$$

The asymptotic expansion then is

$$1-\rho_{k,n}(\mathscr{X})=\frac{\pi^2}{2}k^2n^{-1}-\frac{\pi^3}{3}k^3n^{-3}+\mathcal{O}(n^{-4}), \tag{6.26}$$

giving $\mathscr{X}\in S^{(2,1)}$ and $\mathscr{X}\notin S^{(2,\mu)}$, $\mu\geqslant 2$. Thus this kernel as well is not appropriate for linear combinations (4.5).

7. Linear Combinations which Are Not Saturated

In this section some examples of non-saturated linear combinations shall be briefly presented. These do not fall under the foregoing considerations. To this end consider the *best trigonometric approximation* $E_n(C_{2\pi};f)$ for $f\in C_{2\pi}$, $n\in\mathbb{N}$, given by

$$E_n(C_{2\pi};f):=\inf_{t_n\in\pi_n}\|f-t_n\|=\|f-t_n^*\|. \tag{7.1}$$

In case that for the convergence factors of a kernel $\not{k}$ there holds

$$\rho_{k,n}(\not{k})=1 \qquad (1\leqslant k\leqslant n), \tag{7.2}$$

one immediately has

$$I_n(\not{k};t_n;x)=t_n(x) \qquad (t_n\in\pi_n). \tag{7.3}$$

Thus for $f\in C_{2\pi}$,

$$\begin{aligned}\|I_n(\not{k};f;\cdot)-f(\cdot)\|&\leqslant\|I_n(\not{k};f-t_n^*;\cdot)\|+\|f-t_n^*\|\\&\leqslant(L_n(\not{k})+1)\|f-t_n^*\|.\end{aligned} \tag{7.4}$$

For example, the well-known linear combination $\tau=\{\tau_n\}_{n\in\mathbb{N}}$ of the Fejér kernel $\mathscr{F}$

$$\tau_n(x):=2F_{2n-1}-F_{n-1}(x) \qquad (x\in\mathbb{R})$$

satisfies (7.2), so that $\|I_n(\tau;f;\cdot)-f(\cdot)\|=\mathcal{O}(1)\,E_n(C_{2\pi};f)$ for $f\in C_{2\pi}$.

This method may be generalized in the following way: Suppose the convergence factors of $\not{k}$ admit an expansion ($\tau=1$ or $\tau=2$, $\mu\in\mathbb{N}$)

$$1-\rho_{k,n}(\not{k})=\sum_{j=1}^{\mu}(-1)^{j+1}\psi_j(k)n^{-\tau j}+h_{k,\mu}n^{-\tau(\mu+1/2)}, \tag{7.5}$$

for certain $h_{k,\mu}\in\mathbb{R}$, $1\leqslant k\leqslant n$; thus in particular $\not{k}\in S^{(\tau,\mu)}$. Then a linear combination can be built up similar to (4.5) such that all of the terms on the right side of (7.5) can be cancelled. As an example, take the kernels of Jackson and de La Vallée Poussin $\mathscr{P}^p$ (6.12) with convergence factors (6.16).

For $p=2$ and choosing in the *Ansatz*

$$q_n(x)=\gamma_1 P^2_{a_1 n}(x)+\gamma_2 P^2_{a_2 n}(x)+\gamma_3 P^3_{a_3 n}(x) \qquad (x\in\mathbb{R}), \tag{7.6}$$

$a_i=i$, $1\leqslant i\leqslant 3$, one can uniquely determine the coefficients γ_i so that for $\mathscr{q}=\{q_n\}_{n\in\mathbb{N}}$ there holds (7.2). This yields

COROLLARY 7.1. *The unique linear combination $\mathscr{q}$ of Jackson–de La Vallée Poussin kernels* (7.6) *satisfying* (7.2) *is given by*

$$q_n(x)=\tfrac{1}{12}(27P^2_{3n}(x)-16P^2_{2n}(x)+P^2_n(x)) \qquad (x\in\mathbb{R}). \tag{7.7}$$

Furthermore, there holds for the corresponding singular integral

$$\|I_n(\mathscr{q};f;\cdot)-f(\cdot)\|=\mathcal{O}(1)\,E_n(C_{2\pi},f).$$

Notice, for instance, that the kernels of Jackson $\mathscr{J}^p$ of (6.1), $2\leqslant p\in\mathbb{N}$, admit no expansion of the form (7.5); nevertheless also in this case linear combinations can be constructed that are not saturated. This will be briefly outlined for the parameter $p=2$. The convergence factors are then

$$\rho_{k,n}(\mathscr{J}^2)=\frac{1}{2n(2n^2+1)}\begin{cases}3k^2-6nk^2-3k+2n(2n^2+1), & 0\leqslant k\leqslant n,\\ -k^3+6nk^2-(12n^2-1)k+8n^3-2n, & n\leqslant k\leqslant 2n.\end{cases} \tag{7.8}$$

Thus taking the same *Ansatz* as (7.6), now with $\mathscr{J}^2$ and $a_i=i$, $1\leqslant i\leqslant 3$, this leads to

COROLLARY 7.2. *The unique linear combination $\tilde{\mathscr{q}}$ of Jackson kernels* (7.6) *satisfying* (7.2) *is given by*

$$\tilde{q}_n(x)=\frac{1}{4n^3}\left\{-\frac{n(2n^2+1)}{9}J^2_n(x)+\frac{2n(8n^2+1)}{3}J^2_{2n}(x)\right.$$
$$\left.-n(18n^2+1)\,J^2_{3n}(x)+\frac{4n(32n^2+1)}{9}J_{4n}(x)\right\} \qquad (x\in\mathbb{R}). \tag{7.9}$$

Furthermore, there holds for the corresponding singular integral,

$$\|I_n(\tilde{\mathscr{q}};f;\cdot)-f(\cdot)\|=\mathcal{O}(1)\,E_n(C_{2\pi},f).$$

Notice that now the coefficients depend on n, but since they are bounded $\tilde{\varphi} = \{\tilde{\varphi}_n\}_{n \in \mathbb{N}}$ defines an approximation process, which is not saturated by construction. The above example (7.9) may be found in [30], where, however, the construction is rather complicated. The method presented can be extended to arbitrary $p > 2$ both for Jackson as well as for Jackson–de La Vallée Poussin kernels (but with rather extensive calculations).

All in all, Jackson–de La Vallée Poussin kernels seem more suitable for these linear combinations since the calculation of the coefficients is less elaborate. This is of course due to their Fejér-type structure. Taking linear combinations of B-splines (thus the Fourier transforms of Jackson–de La Vallée Poussin kernels) as suitable kernels for generalized sampling sums in signal analysis has also been carried out; it will be presented in a forthcoming paper.

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